# 2.4 Ratios and Proportions

## RATIOS

A **ratio** between two numbers *compares their relative size*, using division, and this relative size will not change, no matter how large or small the numbers may get.

> The **ratio** of two numbers, A and B, is written as either A **:** B or $\frac{A}{B}$ .
> In words, this is read, "**A to B**."

**Consider this:** Two brothers, Kenny and Jimi decided to go into business together selling lemonade. Kenny, the older brother, contributed $10 to get the business started and Jimi contributed $6. When the profits were counted at the end of the summer, Kenny got $75 and Jimi got $45. Is this a fair sharing of the profits?

To answer this question, look at the relative size, based on the ratio **Kenny : Jimi**, of

(1) the investments each made: $10 **:** $6 or $\frac{10}{6}$

and (2) the profits each received: $75 **:** $45 or $\frac{75}{45}$

*If these two ratios are equivalent, then the share of the profits is fair.*
One way we can see if the ratios are equivalent is to simplify each and see if they simplify to the same ratio or fraction:

$\frac{10}{6}$ simplifies by a factor of 2 to $\frac{5}{3}$

$\frac{75}{45}$ simplifies by a factor of 15 to $\frac{5}{3}$

Since each fraction simplifies to $\frac{5}{3}$ , it is safe to conclude that the sharing of profits was fair.

A second way to check if the ratios are equivalent is to set them next to each other and *cross multiply* (the question mark over the equal sign means we're trying to find out if the fractions are equivalent or not):

$$\frac{10}{6} \overset{?}{=} \frac{75}{45}$$

Set the two fractions next to each other.

$$\frac{10}{6} \overset{?}{=} \frac{75}{45}$$

Cross multiply.

(Multiply the denominators across the equal sign; the numerators remain where they are.)

$$10 \cdot 45 \overset{?}{=} 6 \cdot 75$$

$$450 = 450$$

If, after cross multiplying, the products are not the same, then we say that the ratios are not equivalent.

**Example 1:** Determine whether or not the ratios are equivalent by <u>cross multiplying</u>.

a) $\frac{6}{4} \overset{?}{=} \frac{15}{10}$    b) $\frac{8}{20} \overset{?}{=} \frac{12}{25}$

**Procedure:** Multiply the denominators, across the equal sign, to the numerators on the other side.

**Answer:**

a) $\frac{6}{4} \overset{?}{=} \frac{15}{10}$    b) $\frac{8}{20} \overset{?}{=} \frac{12}{25}$

$\frac{6}{4} \overset{?}{\times} \frac{15}{10}$    $\frac{8}{20} \overset{?}{\times} \frac{12}{25}$

$6 \cdot 10 \overset{?}{=} 4 \cdot 15$    $8 \cdot 25 \overset{?}{=} 20 \cdot 12$

$60 = 60$    $200 \neq 240$

So, $\frac{6}{4}$ and $\frac{15}{10}$ ***are*** equivalent.    So, $\frac{8}{20}$ and $\frac{12}{25}$ are ***not*** equivalent.

**Exercise #1:** Determine whether or not the ratios are equivalent by cross multiplying. Use Example 1 as a guide.

a) $\frac{6}{15} \overset{?}{=} \frac{8}{25}$    b) $\frac{9}{6} \overset{?}{=} \frac{12}{8}$    c) $\frac{9}{12} \overset{?}{=} \frac{15}{20}$

## PROPORTIONS

When we set two equivalent ratios equal to each other we form an equation, called a **proportion**.

**Exercise #2:** Based on the definition of *proportion*, above, which of these are proportions? (Refer to Exercise 1.)

a) $\frac{6}{15} \overset{?}{=} \frac{8}{25}$    b) $\frac{9}{6} \overset{?}{=} \frac{12}{8}$    c) $\frac{9}{12} \overset{?}{=} \frac{15}{20}$

When a proportion has a variable within it—maybe in one or both of the numerators or one or both of the denominators—we can solve for that variable by cross multiplying.

Examples of *proportion equations* are $\frac{w}{8} = \frac{27}{12}$, $\frac{20}{x} = \frac{12}{15}$ and $\frac{x}{5} = \frac{x + 16}{15}$.

Of course, each proportion equation has fractions, so it's common for us to solve by clearing the fractions, and cross multiplying allows us to do that:

**Clearing the Fractions of a Proportion by Cross Multiplying**

| | |
|---|---|
| $\frac{a}{b} = \frac{c}{d}$ | Here is the proportion. |
| $\frac{a}{b} \times \frac{c}{d}$ | Cross multiplying (multiplying across the equal sign). |
| $a \cdot d = b \cdot c$ | We have cleared the fractions and have a new equation to solve, one without fractions. |

**Caution:** If the equation involves <u>more than just two equal fractions</u>, then it is *not* a proportion and we *cannot* use cross multiplication to clear the fractions. Here are some examples of equations with fractions and some examples of proportions.

| **Equations with fractions** | **Proportions** |
|---|---|
| $\frac{x}{4} + 7 = \frac{x}{3}$ | $\frac{35}{10} = \frac{x}{4}$ |
| $\frac{x + 3}{8} - \frac{x}{2} = 5$ | $\frac{x + 2}{4} = \frac{x - 1}{2}$ |

↑

We ***cannot*** use cross multiplication to solve these.

**Example 2:** Solve for x in each proportion.

a) $\frac{x}{8} = \frac{3}{4}$ b) $\frac{5}{4} = \frac{10}{x}$

**Procedure:** Use cross multiplication to get the equation into a more familiar form. Then solve the resulting equation.

**Answer:**

a)
$$\frac{x}{8} = \frac{3}{4}$$
$$x \cdot 4 = 8 \cdot 3$$
$$4x = 24$$
$$\frac{4x}{4} = \frac{24}{4}$$
$$x = 6$$

b)
$$\frac{5}{4} = \frac{10}{x}$$
$$5 \cdot x = 4 \cdot 10$$
$$5x = 40$$
$$\frac{5x}{5} = \frac{40}{5}$$
$$x = 8$$

**Exercise #3:** Solve for x in each proportion. Use Example 2 as a guide.

a) $\frac{x}{12} = \frac{10}{15}$ b) $\frac{10}{4} = \frac{15}{x}$

c) $\frac{4}{18} = \frac{x}{27}$ d) $\frac{20}{x} = \frac{15}{9}$

If one ratio has no variable (is only numerical), then it might be able to be simplified before cross multiplying. Simplifying first is helpful because the numbers are easier to work with.

**Example 3:** Solve for x in each proportion.

a) $\frac{21}{12} = \frac{x}{8}$   b) $\frac{10}{x} = \frac{25}{30}$

**Procedure:** Simplify one of the ratios, if possible. Use cross multiplication to get the equation into a more familiar form.

**Answer:**

| | a) $\frac{21}{12} = \frac{x}{8}$ | b) $\frac{10}{x} = \frac{25}{30}$ |
|---|---|---|
| **First:** | Simplify $\frac{21}{12}$ by a factor of 3 | Simplify $\frac{25}{30}$ by a factor of 5 |
| Now cross multiply and solve: | $\frac{7}{4} = \frac{x}{8}$ | $\frac{10}{x} = \frac{5}{6}$ |
| | $7 \cdot 8 = 4 \cdot x$ | $10 \cdot 6 = x \cdot 5$ |
| | $56 = 4x$ | $60 = 5x$ |
| | $\frac{56}{4} = \frac{4x}{4}$ | $\frac{60}{5} = \frac{5x}{5}$ |
| | $14 = x$ | $12 = x$ |
| | $x = 14$ | $x = 12$ |

**Exercise #4** Solve for x in each proportion by first simplifying the numerical ratio. Use Example 3 as a guide.

a) $\frac{8}{60} = \frac{x}{45}$

b) $\frac{18}{x} = \frac{15}{10}$

c) $\frac{x}{35} = \frac{24}{40}$

d) $\frac{30}{12} = \frac{20}{x}$

Sometimes proportions have unknown information in more than one place. For example, the proportion equation $\frac{x+6}{5} = \frac{x}{2}$ can still be solved by cross multiplying.

First, though, we should recognize that—because the division bar is a grouping symbol—the left side numerator is a quantity:

$$\frac{(x+6)}{5} = \frac{x}{2}$$ Now cross multiply.

$$(x+6)\cdot 2 = 5\cdot x$$

As we cross multiply, the parentheses remain until we distribute:

$$2x + 12 = 5x$$

Now we solve as we do any linear equation—by isolating the variable.

This process remains true even if both variable terms are in the denominator.

**Example 4:** Solve for x in each proportion. *Check the answer*.

a) $\frac{x+6}{5} = \frac{x}{2}$ b) $\frac{4}{x-1} = \frac{6}{x+2}$

**Procedure:** Use parentheses to group all quantities; then use cross multiplication to get the equation into a linear form. Isolate the variable by solving the linear equation.

**Answer:**

a) $\frac{x+6}{5} = \frac{x}{2}$

Show parentheses: $\frac{(x+6)}{5} = \frac{x}{2}$

Cross multiply: $(x+6)\cdot 2 = 5\cdot x$

Isolate the variable:

$$\begin{array}{rcr} 2x + 12 & = & 5x \\ -2x \quad & = & -2x \\ \hline 12 & = & 3x \end{array}$$

$$\frac{12}{3} = \frac{3x}{3}$$

$$4 = x$$

$$x = 4$$

**Check:**

$$\frac{4+6}{5} \stackrel{?}{=} \frac{4}{2}$$

$$\frac{10}{5} \stackrel{?}{=} 2$$

$2 = 2$ **true!**

b) $\frac{4}{x-1} = \frac{6}{x+2}$

Show parentheses: $\frac{4}{(x-1)} = \frac{6}{(x+2)}$

Cross multiply: $4\cdot(x+2) = (x-1)\cdot 6$

Isolate the variable:

$$\begin{array}{rcr} 4x + 8 & = & 6x - 6 \\ -4x \quad & = & -4x \quad \\ \hline 8 & = & 2x - 6 \\ +6 & = & +6 \\ \hline 14 & = & 2x \end{array}$$

$$\frac{14}{2} = \frac{2x}{2}$$

$$7 = x$$

$$x = 7$$

**Check:**

$$\frac{4}{7-1} \stackrel{?}{=} \frac{6}{7+2}$$

$$\frac{4}{6} \stackrel{?}{=} \frac{6}{9}$$

$\frac{2}{3} = \frac{2}{3}$ **true!**

**Exercise 5:** Solve each proportion. *Check your answer.*

a) $\frac{x + 3}{5} = \frac{x}{2}$

b) $\frac{w}{8} = \frac{w + 12}{24}$

c) $\frac{2x - 2}{4} = \frac{3x - 9}{2}$

d) $\frac{2y - 4}{2} = \frac{3y}{5}$

e) $\frac{2p + 7}{4p} = \frac{3}{4}$

f) $\frac{5}{8} = \frac{2c + 2}{4c - 4}$

To this point, we've worked with only whole number values. However, many times a ratio will include fractional values in the numerator or denominator.

For example, if it takes $\frac{2}{3}$ cups of milk for 4 pancakes, then we can double the recipe to make 8 pancakes, but it will take $\frac{4}{3}$ cups of milk.

This can be seen in the proportion with ratios milk **:** pancakes. $\dfrac{\frac{2}{3}}{4} = \dfrac{\frac{4}{3}}{8}$

It might also be that $\frac{4}{3}$ is expressed as a mixed number, $1\frac{1}{3}$ : $\dfrac{\frac{2}{3}}{4} = \dfrac{1\frac{1}{3}}{8}$

**Example 5:** Solve for x in each proportion. Write any improper fraction *answer* as a mixed number.

a) $\frac{2}{x} = \frac{8}{3}$ b) $\dfrac{1\frac{3}{5}}{4} = \dfrac{x}{5}$ c) $\frac{3}{x} = \frac{4}{4x-6}$

**Procedure:** Use cross multiplication to get the equation into a linear form.

**Answer:**

**a)** $\frac{2}{x} = \frac{8}{3}$

$6 = 8x$

$\frac{6}{8} = \frac{8x}{8}$

$\frac{3}{4} = x$

$\boxed{\mathbf{x = \frac{3}{4}}}$

**b)** $\dfrac{1\frac{3}{5}}{4} = \dfrac{x}{5}$

Write $1\frac{3}{5}$ as an improper fraction → $\dfrac{\frac{8}{5}}{4} = \dfrac{x}{5}$

Cross multiply: → $\frac{8}{5} \cdot 5 = 4x$

$8 = 4x$

$2 = x$

$\boxed{\mathbf{x = 2}}$

**c)** $\frac{3}{x} = \frac{4}{4x-6}$

$3(4x-6) = 4x$

$12x - 18 = 4x$

$\underline{-12x \qquad = -12x \qquad .}$

$-18 = -8x$

$\frac{-18}{-8} = \frac{-8x}{-8}$

$\frac{9}{4} = x$

Write this answer as a mixed number.

$\boxed{\mathbf{x = 2\frac{1}{4}}}$

**Exercise 6:** Solve each proportion. Write any improper fraction *answer* as a mixed number.

a) $\frac{x}{4} = \frac{1}{6}$

b) $\frac{15}{2} = \frac{6}{w}$

c) $\frac{3x + 2}{30} = \frac{x}{5}$

d) $\frac{3}{y} = \frac{36}{6y + 5}$

e) $\frac{2\frac{2}{3}}{2} = \frac{p}{3}$

f) $\frac{5}{3\frac{3}{4}} = \frac{8}{m}$

# Answers to each Exercise

## Section 2.4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Exercise #1:** | a) | $150 \neq 120$ not equivalent | b) | $72 = 72$ equivalent | c) | $180 = 180$ equivalent | | |
| **Exercise #2:** | a) | not a proportion | b) | is a proportion | c) | is a proportion | | |
| **Exercise #3:** | a) | $x = 8$ | b) | $x = 6$ | c) | $x = 6$ | d) | $x = 12$ |
| **Exercise #4:** | a) | $x = 6$ | b) | $x = 12$ | c) | $x = 21$ | d) | $x = 8$ |
| **Exercise #5:** | a) | $x = 2$ | b) | $w = 6$ | c) | $x = 4$ | d) | $y = 5$ |
| | e) | $p = 7$ | f) | $c = 9$ | | | | |
| **Exercise #6:** | a) | $x = \frac{2}{3}$ | b) | $w = \frac{4}{5}$ | c) | $x = \frac{2}{3}$ | d) | $y = \frac{5}{6}$ |
| | e) | $p = 4$ | f) | $m = 6$ | | | | |

## Section 2.4 Focus Exercises

**1.** Use cross multiplication to determine whether the pair of fractions is equivalent or not.

a) $\frac{18}{12} \overset{?}{=} \frac{3}{2}$

b) $\frac{15}{6} \overset{?}{=} \frac{5}{2}$

c) $\frac{4}{12} \overset{?}{=} \frac{5}{10}$

d) $\frac{8}{10} \overset{?}{=} \frac{6}{8}$

e) $\frac{12}{30} \overset{?}{=} \frac{8}{15}$

f) $\frac{18}{24} \overset{?}{=} \frac{6}{9}$

g) $\frac{20}{30} \overset{?}{=} \frac{8}{12}$

h) $\frac{25}{15} \overset{?}{=} \frac{10}{6}$

**2.** Solve for the variable in each proportion.

a) $\frac{2}{3} = \frac{x}{6}$

b) $\frac{5}{10} = \frac{3}{c}$

c) $\frac{x}{6} = \frac{21}{14}$

d) $\frac{15}{p} = \frac{30}{4}$

e) $\frac{\frac{1}{2}}{x} = \frac{1}{8}$

f) $\frac{\frac{1}{2}}{x} = \frac{1}{\frac{4}{9}}$

g) $\frac{x + 1}{5} = \frac{3}{15}$

h) $\frac{x + 3}{12} = \frac{7}{6}$

i) $\frac{11}{7} = \frac{4 - 3y}{14}$

j) $\frac{x + 6}{5} = \frac{x - 3}{2}$

**3.** Solve each proportion. Write any improper fraction *answer* as a mixed number.

a) $\frac{3}{14} = \frac{x}{21}$

b) $\frac{1}{p} = \frac{10}{8}$

c) $\frac{5\frac{1}{2}}{m} = \frac{11}{6}$

d) $\frac{x}{3\frac{1}{8}} = \frac{8}{5}$

e) $\frac{5x + 2}{30} = \frac{x}{3}$

f) $\frac{4}{x} = \frac{16}{2x + 7}$